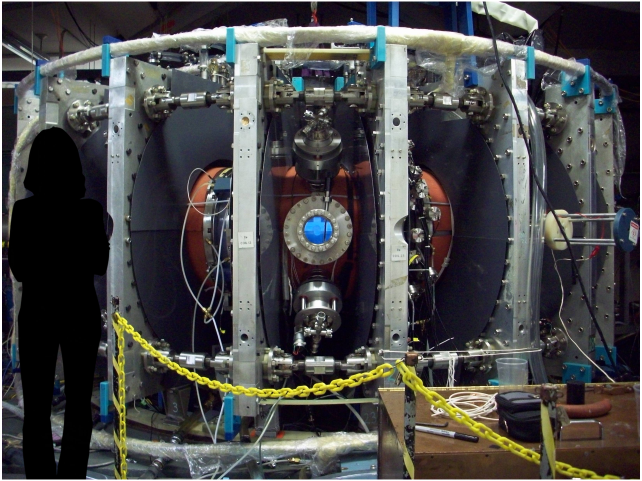
Magnetic Control of Perturbed Plasma Equilibria

Nikolaus Rath

February 17th, 2012

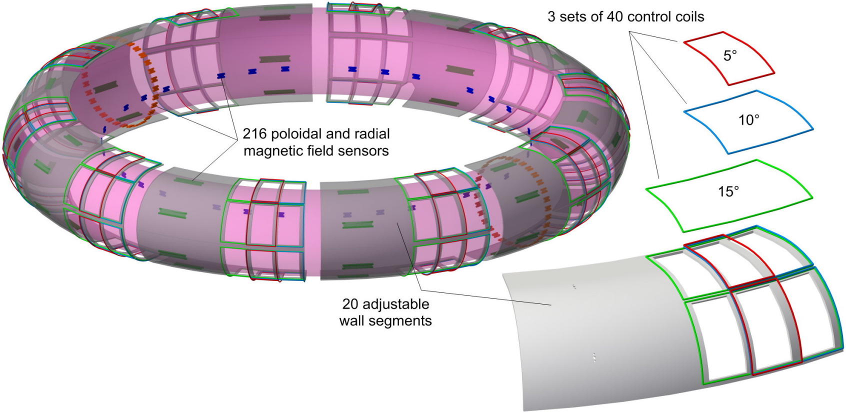
Columbia University
in the City of New York

The HBT-EP Tokamak



S. Angelini, J. Bialek, P. Byrne, B. DeBono, P. Hughes, J. Levesque, B. Li, M. Mauel, G. Navratil, Q. Peng, D. Rhodes, D. Shiraki, C. Stoafer

Recent upgrade allows high-resolution magnetic measurement and control



Inside the HBT-EP vacuum chamber

Plasma dynamics are complicated

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}_\alpha \cdot \nabla f_\alpha - e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \cdot \frac{\partial f_\alpha}{\partial \vec{p}} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi \vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{E} = 4\pi\rho, \quad \nabla \cdot \vec{B} = 0$$

$$\rho = e \int (f_i - f_e)\, d^3\vec{p}$$

$$\vec{j} = e \int (f_i - f_e)\vec{v}\, d^3\vec{p}$$

$$\vec{v}_\alpha = \frac{\vec{p}/m_\alpha}{\sqrt{1 + p^2/m_\alpha^2 c^2}}$$

MHD makes things easier

With a few not-quite-appropriate approximations:

$$\rho \frac{d\vec{u}}{dt} = (\nabla \times \vec{B}) \times \vec{B} - \nabla p \qquad \frac{d\rho}{dt} = -\nabla \cdot (\rho \vec{u}) \tag{1}$$

$$\frac{d\vec{B}}{dt} = \nabla \times (\vec{u} \times \vec{B}) \qquad \frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{2}$$

But even better, in a Tokamak:

$$\|\vec{u}\| \approx \frac{\|\vec{B}\|}{\sqrt{\mu_0 \rho}} \approx \frac{\text{meters}}{\text{microseconds}} \tag{3}$$

$$\Rightarrow \quad \frac{d\vec{u}}{dt} \approx 0 \tag{4}$$

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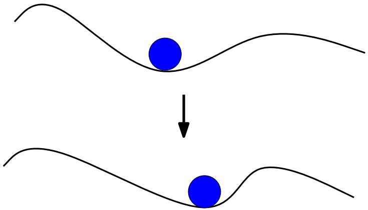
Tokamak plasmas transition through MHD equilibria



- Plasma reaches force-balance within microseconds
- Longer lived plasmas must thus be in equilibrium
- Evolution is caused by all our approximations
- The plasma always has minimum MHD energy, but the properties of the minimum-energy-state are changing slowly

Tokamak equilibria are tractable

This talk is about:

$$(\nabla \times \vec{B}) \times \vec{B} = \nabla p \qquad (5)$$

In Tokamaks, the plasma state is axisymmetric with a 3d perturbation

- Solving $(\nabla \times \vec{B}) \times \vec{B} = \nabla p$ in axisymmetry is easy – so we assume someone did it.
- Solving $(\nabla \times \vec{B}) \times \vec{B} = \nabla p$ in 3d is hard, so we don't want to do it.
- So will solve $(\nabla \times \vec{B}) \times \vec{B} = \nabla p$ with just small 3d perturbation - Tokamaks are great!
- A small 3d displacement $\vec{\zeta}$ must minimize (Boozer, 1999 & 2003)

$$dW(\vec{\zeta}) = \int_{\text{plasma}} \vec{\zeta} \cdot \vec{F}(\vec{\zeta})\, dV \tag{6}$$

- $\vec{F}$ is known from the axisymmetric solution, minimization is just number crunching.
- Just need boundary conditions on plasma surface

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HBT-EP

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The permeability matrix defines the plasma response to external fields

- The plasma shape follows magnetic field lines
- The non-axisymmetric field on the plasma surface is the boundary conditions for the plasma displacement.
- The field has components from both the plasma and the environment, write it as

$$b(\vec{\zeta}) = b_{\text{plasma}} + b_{\text{external}} \tag{7}$$

- For easier math, write surface functions as vectors of expansion coefficients, e.g.

$$b(\theta,\phi) = \sum_i \Phi_i f_i(\theta,\phi) = \vec{\Phi}\cdot\vec{f}(\phi,\theta) \tag{8}$$

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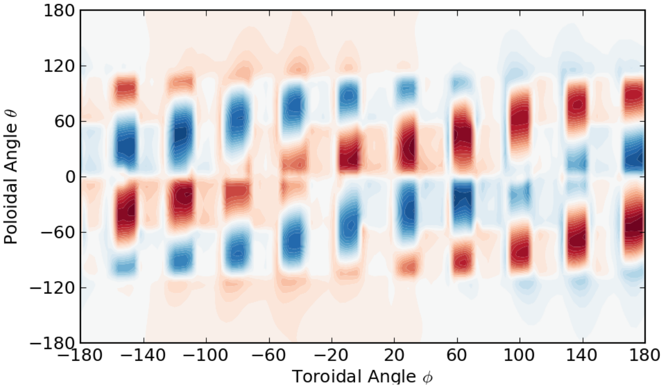
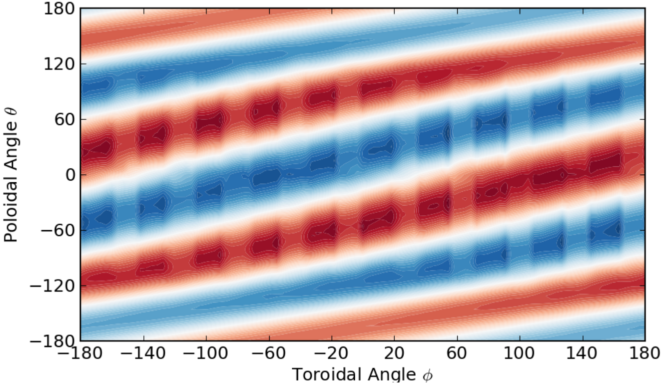
Plasma response example

External field



Plasma response example

Total field



HBT-EP

Half-Time Summary

- We want to control the plasma
- In Tokamaks, the plasma is mostly in MHD equilibrium
- In Tokamaks, the equilibrium is also approximately axisymmetric
- The axisymmetric part is constant, the 3d perturbation changes in time
- The axisymmetric part is easily measured and calculated, we assume it is known
- The 3d perturbation is uniquely defined by magnetic field due to external currents on the plasma surface

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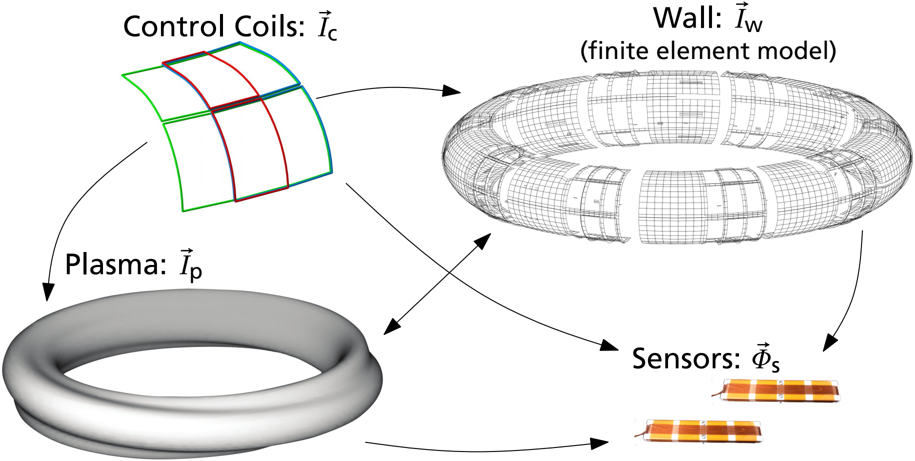
The system model consists of plasma, wall, control coils, and sensors



External Currents $\vec{I}_x = (\vec{I}_c, \vec{I}_x)$

Maxwell's equations allow to represent all system components as vectors

- Maxwell: the normal magnetic field on a closed surface uniquely defines the external field in the enclosed volume
- Plasma sees all external currents as $\vec{\Phi}_x = \mathbf{M}_{px}.\vec{I}_x$
- External circuits see plasma as $\vec{\Phi}_p = (\mathbf{P} - \mathbf{1}).\vec{\Phi}_x$.
- We can express all interactions as matrices. The complete system obeys

$$\mathbf{R}_x.\vec{I}_x + \left[\mathbf{L}_x + \mathbf{M}_{xp}.\mathbf{L}_p^{-1}.(\mathbf{P} - \mathbf{1}).\mathbf{M}_{px}\right].\frac{d\vec{I}_x}{dt} = \vec{V}_x \qquad (10)$$

- **M** and **L** matrices encapsulate all geometric information.
- **P** depends on the axisymmetric equilibrium and contains information about the plasma.

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Controlling the plasma

To control the solution of

$$(\nabla \times \vec{B}) \times \vec{B} = \nabla p \tag{11}$$

we need to control a system of the form

$$\frac{d\vec{I}_x}{dt} = \mathbf{A}.\vec{I}_x + \mathbf{B}.\vec{V}_x \tag{12}$$

Required steps:

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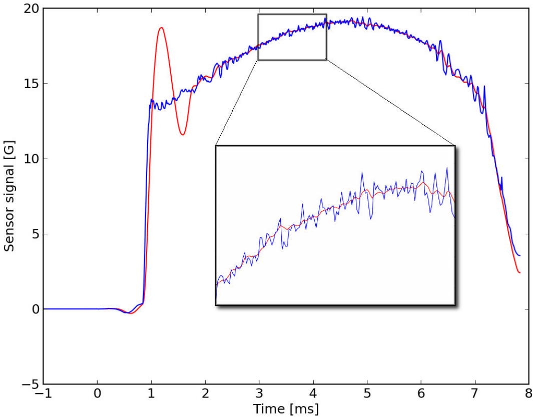
(not discussed in this talk)

Equilibrium fields are subtracted using continuous polynomial prediction

- Measurement errors in the (large) axisymmetric contribution easily exceed non-axisymmetric fluctuations.
- Solution: Continuously fit last *n* measurements of every sensor to quadratic polynomial, subtract predicted amplitude.



SVD of measurement matrix determines relative visibility of states

- Sensors measure magnetic flux:

$$\begin{aligned}\vec{\Phi}_\mathrm{s} &= \mathbf{M}_\mathrm{sx}.\vec{I}_\mathrm{x} + \mathbf{M}_\mathrm{sp}.\vec{I}_\mathrm{p} \\ &= \left[\mathbf{M}_\mathrm{sx} + \mathbf{M}_\mathrm{sp}.\mathbf{L}_\mathrm{p}^{-1}.(\mathbf{P}-\mathbf{1}).\mathbf{M}_\mathrm{px}\right].\vec{I}_\mathrm{x} =: \mathbf{C}.\vec{I}_\mathrm{x}\end{aligned} \tag{13}$$

- Singular value decomposition of **C** gives

$$\mathbf{U}.\mathbf{S}.\mathbf{V}^\dagger = \begin{pmatrix} \uparrow & \uparrow & \\ \vec{u}_1 & \vec{u}_2 & \cdots \\ \downarrow & \downarrow & \end{pmatrix} . \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \end{pmatrix} . \begin{pmatrix} \longleftarrow & \vec{v}_1 & \longrightarrow \\ \longleftarrow & \vec{v}_2 & \longrightarrow \\ & \vdots & \end{pmatrix} \tag{14}$$

- $\vec{v}_i$ is the basis of directly measurable states
- s_i is the relative visibility of a state
- We could drop invisible states and invert, but there is a better solution

The system model is used to estimate the state

- Control system continuously calculates:

$$\frac{d\vec{I}_{\mathrm{x}}}{dt} = \mathbf{A}.\vec{I}_{\mathrm{x}} + \mathbf{B}.\vec{V}_{\mathrm{x}} + \mathbf{K}.\left(\vec{\Phi}_{\mathrm{s}} - \mathbf{C}.\vec{I}_{\mathrm{x}}\right) \tag{15}$$

- Error in calculated $\vec{I}_{\mathrm{x}}$ obeys

$$\frac{d\vec{e}}{dt} = (\mathbf{A}^{\mathrm{T}} - \mathbf{C}^{\mathrm{T}}.\mathbf{K}^{\mathrm{T}}).\vec{e} \tag{16}$$

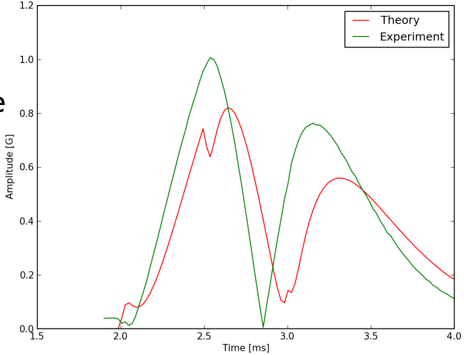
- Could pick **K** to obtain specific eigenvalues – but which exactly?
- Better choice: let **R** be measurement uncertainties and **S** mode visibilities, and choose **K** to minimize

$$\int_0^\infty \|\mathbf{S}.\vec{e}(t)\|^2 + \|\mathbf{R}.\mathbf{K}^{\mathrm{T}}.\vec{e}\|^2 \, dt \tag{17}$$

- More visible states will converge faster, more reliable sensors will be used preferentially

Experimental results: wall model and shot reproducibility needs to be improved

- HBT-EP plasmas are short-lived, so equilibrium needs to be computed offline
- Variations in major radius change equilibrium considerably between shots, preventing pre-computation
- Plan: address by extending control to plasma major radius



- Vacuum testing show reasonably agreement, better wall model may give further improvements. Work in progress to add vacuum vessel.

Summary

1. The evolution of Tokamak plasmas can be approximated as a transition through a sequence of MHD equilibria.
2. These equilibria are approximately axisymmetric and consist of a static, axisymmetric part and a changing, 3-dimensional perturbation.
3. Transitions caused by changing external fields can be modeled as a linear, time invariant system. This system can be controlled with standard techniques from control theory, using magnetic sensors and control coils.
4. Experimental tests show reasonable agreement for vacuum model. Plasma tests were impaired by low shot reproducibility, which will be addressed by control of plasma major radius.