Control of linear modes in cylindrical resistive MHD with a resistive wall, plasma rotation, and complex gain

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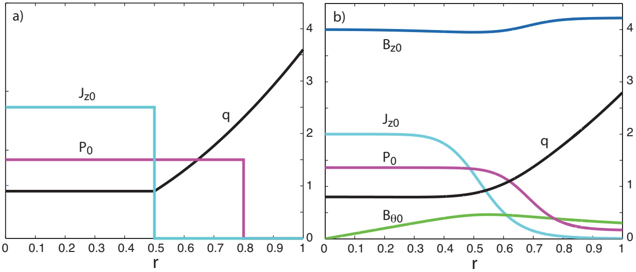
Outline

- Final step before full toroidal, resistive MHD control study: Linear cylindrical tokamak model with finite β for control of **resistive-plasma resitive-wall modes**, using a combination of normal *and* tangential magnetic field measurements
- Comparison of Full Resistive MHD with diffuse profiles compared with Analytic Reduced MHD in step function profiles, facilitates understanding of the origin physics results
- Marginal stability values $\beta_{rp,rw} < \beta_{rp,iw} < \beta_{ip,rw} < \beta_{ip,iw}$ (resistive or ideal, plasma or wall) indicate transition points
- Imaginary gain $\sim$ plasma rotation stabilizes below $\beta_{rp,iw}$ because rotation supresses the diffusion of flux from the plasma out through the wall
- More surprisingly, imaginary gain or rotation **destabilizes** above $\beta_{rp,iw}$ because it prevents the feedback flux from entering the plasma through the resistive wall.
- Method of using complex gain to optimize in the presence of rotation for $\beta > \beta_{rp,iw}$.

Full MHD Computation compared with Reduced MHD Analytic model for Intuition

Model 1: Reduced MHD	Model 2: Full MHD
stepfunction profiles $j_{z0}(r) = 2\Theta(a_1 - r)$ and $p_0(r) = p_0(0)\Theta(a_2 - r)$.	smooth profiles $j_{z0}(r) = \frac{2}{(1+(r/a_1)^8)^{5/4}}$ from Furth, Rutherford, Selberg (Flattened) and $p_0(r) = \frac{p_{00}}{(1+(r/a_2)^{16})^{9/8}}$
Ideal outer region, Tearing layers either RI regime $(\gamma_d\tau_t)^{5/4} - \Delta_1$ or VR regime $\gamma_d\tau - \Delta_1$	Plasma resistivity, viscosity constant: $S = 10^5 - 10^8$, $P = 0.01$
$a_1 < r_t \lesssim a_2$ Pressure drive outside r_t enhances wall interaction; (somewhat mimics toroidal external kink)	
Resistive wall by thin wall approximation. $\gamma\tau_w\tilde{\psi}(r_w) = [\tilde{\psi}']_{r_w}$	
Feedback by control equation. $\tilde{\psi}(r_c) = -G\tilde{\psi}(r_w) + K\tilde{\psi}'(r_w-)$	
Rotation by Doppler shift. $\gamma_d \equiv \gamma + i\Omega$.	

Equilibrium Models are comparable



Model 1: Reduced MHD	Model 2: Full MHD
$a_1 = 0.5$, $a_2 = 0.8$, $r_w = 1$, $r_c = 1.5$, $q(0) = 0.9$	$a_1 = 0.55$, $a_2 = 0.7$, $r_w = 1$, $r_c = 1.5$, $q(0) = 0.8$.
$q(r) = q(0)$ for $r < a_1$ and $q(r) = q(0)\frac{r^2}{a_1^2}$ for $r > a_1$ where $q(0) = B_0/R$ and R is the major radius	$B_z(r)$ from radial force balance $j_{\theta 0}B_{z0} - j_{z0}B_{\theta 0} = p_0'(r)$ by $\frac{B_{z0}^2}{2} = \frac{B_0^2}{2} + p_{00} - p_0(r) - \int_0^r j_{z0}(r')B_{\theta 0}(r')dr'$. $B_0 = B_{z0}(0)$ specifies $q(0)$

Model 1: Analysis simplified to a matching problem between regions

Outer region

$$\nabla_\perp^2 \tilde{\psi} = \frac{m j_{z0}'(r)}{rF(r)} \tilde{\psi} + \frac{2m^2 B_{\theta 0}^2(r) p_0'(r)}{B_0^2 r^3 F(r)^2} \tilde{\psi}$$

$$= -A\delta(r-a_1)\tilde{\psi} - B\delta(r-a_2)\tilde{\psi}$$

Solution can be expressed by a basis set

$$\tilde{\psi}(r) = \alpha_1 \psi_1(r) + \alpha_2 \psi_2(r) + \alpha_3 \psi_3(r)$$

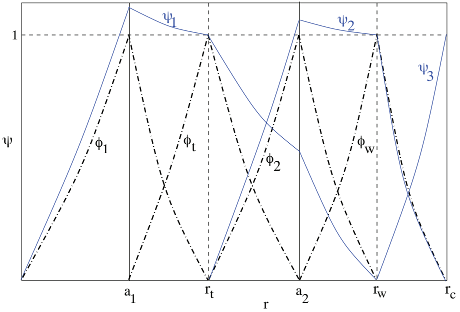
With matching conditions at

$$\gamma_d \tau_t \tilde{\psi}(r_t) = \left[\tilde{\psi}'\right]_{r_t} \quad \text{Tearing layer}$$

$$\gamma \tau_w \tilde{\psi}(r_w) = \left[\tilde{\psi}'\right]_{r_w} \quad \text{Resistive wall}$$

$$\tilde{\psi}(r_c) = -G\tilde{\psi}(r_w) + K\tilde{\psi}'(r_w-) \quad \text{Feedback}$$

Model 1: Basis functions allow for analytic solution construction



- Basis function method (from early Culham years?) of separating the solution into zones with superimposed solutions shielded from neighboring resonant surfaces or conducting walls.
- Further separating the solution into the plasma response (ψ_1) and the resistive wall / control coil external solution (ψ_2) simplifies the analysis into a 2x2 matrix structure for coefficients of ψ_1 and ψ_2. (ψ_3, the control flux then determined)

Model 1 : Simple 2x2 Matrix Structure offers Intuitive Understanding

For VR regime

$$\begin{pmatrix} \Delta_1 - \gamma_d \tau_t & l_{21} \\ l_{12} - K l_{32} l_{12} & \Delta_2 - \gamma \tau_w - G l_{32} + K l_{32} l_{22}^{(-)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

VR: $\tau_t \sim S^{2/3}$ for $Pr = 1$.
$\Delta_1 = 0$ is at $\beta_{rp,iw}$
$\Delta_2 = 0$ is at $\beta_{ip,rw}$.
Equivalence of wall rotation and G_i (Finn-Chacon 2004)

For RI regime $\tau_t \sim S^{3/5}$ and $\gamma_d \tau_t \rightarrow (\gamma_d \tau_t)^{5/4}$

For VR or RI can in principle stabilize up to $\beta_{ip,iw}$ ($\Delta_1 = \Delta_2 = \infty$) using both G and K.

Model 2: Full MHD model includes finite β, compressibility, parallel dynamics, resistivity, viscosity

$$\gamma\tilde{\mathbf{v}} = \left(\nabla \times \tilde{\mathbf{B}}\right) \times \mathbf{B}_0 + \mathbf{j}_0 \times \tilde{\mathbf{B}} - \nabla\tilde{p} + \nu\nabla^2\tilde{\mathbf{v}}$$

$$\gamma\tilde{\mathbf{B}} = \nabla \times \left[\tilde{\mathbf{v}} \times \mathbf{B}_0 - \eta\nabla \times \tilde{\mathbf{B}}\right]$$

$$\gamma\tilde{p} = -\tilde{\mathbf{v}} \cdot \nabla p_0 - \Gamma p_0 \nabla \cdot \tilde{\mathbf{v}}$$

Finite difference discretization (with variable grid density) leads to the standard matrix form:

$$\gamma \begin{pmatrix} \tilde{v} \\ \tilde{B} \\ \tilde{p} \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} \tilde{v} \\ \tilde{B} \\ \tilde{p} \end{pmatrix}$$

Model 2: Boundary Conditions in Full MHD: Resistive Wall and Control

Boundary conditions at resistive wall include effect of control coil and complex gain, equivalent to reduced model

$$\gamma_d \tilde{B}_r(r_w) = ik \cdot B_0 \tilde{v}_r$$

$$im\tilde{v}_r/r + r\partial_r(\tilde{v}_\theta/r) = 0$$

$$ik\tilde{v}_r + \partial_r \tilde{v}_z = 0$$

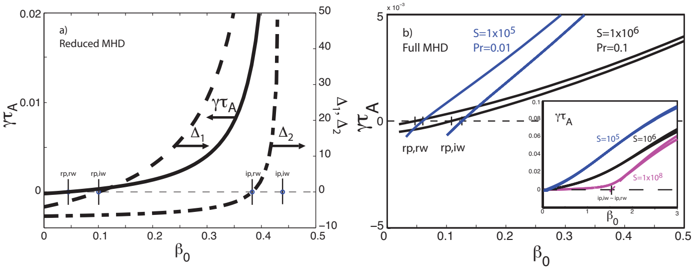
$$\partial_r(r\tilde{B}_\theta) - im\tilde{B}_r = 0$$

$$\gamma_d \tilde{p} = -\tilde{v}_r \partial_r p_0(r_w) - \Gamma p_0(r_w)(\nabla \cdot v)_{r_w}$$

$$\gamma \tau_w \tilde{B}_r = [\tilde{B}_r']_{r_w}$$

$$\tilde{B}_r(r_c) = [-(Gr_w - K)\tilde{B}_r(r_w) + Kr_w \tilde{B}_r{}'(r_w-)]/r_c$$

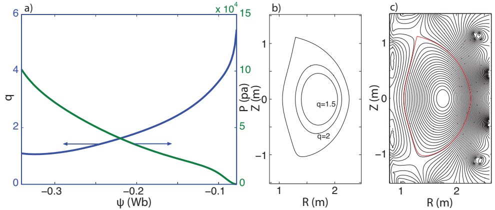
Results: γ vs. β showing $\beta_{rp,rw} < \beta_{rp,iw} < \beta_{ip,rw} \lessapprox \beta_{ip,iw}$; analytic and numerical

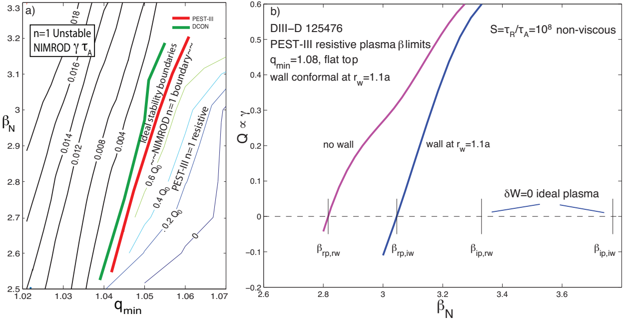


- Growth rate γ for the analytic model (a), with $\beta_{rp,rw} = 0.045$, $\beta_{rp,iw} = 0.101$, $\beta_{ip,rw} = 0.383$, $\beta_{ip,iw} = 0.440$. At $\beta_{rp,iw}$, Δ_1 equals zero and at $\beta_{ip,rw}$, Δ_2 equals zero.
- Numerical results in (b), showing $\beta_{rp,rw} = 0.04$ $\beta_{rp,iw} = 0.11$, and $\beta_{ip,rw} \lesssim \beta_{ip,iw}$.
- Large ideal limits are due to diffuse profiles (computaitonally advantageous), while the focus is on the lower limits $\beta_{rp,rw}$ and $\beta_{rp,iw}$.

In toroidal (DIII-D) configurations the upper limits at far lower β, limits in same order

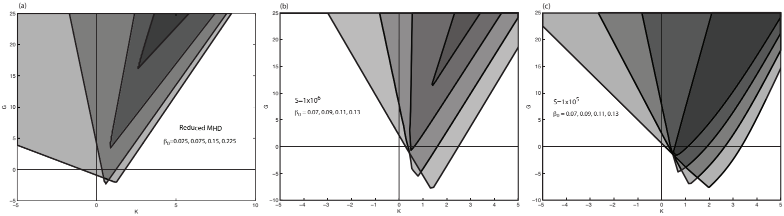
Resistive / ideal plasma limits with / without a (perfectly) conducting wall indicates (without rotation or control) four β limits.





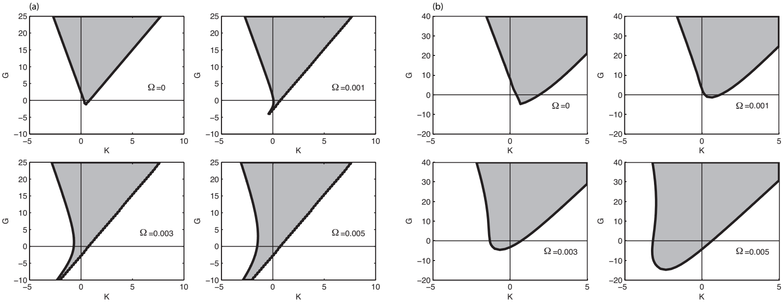
Next steps: resistive wall and control in toroidal. For now: cylindrical.

G-K Maps with $\Omega = G_i = K_i = 0$ Show similar structure between reduced and full MHD models



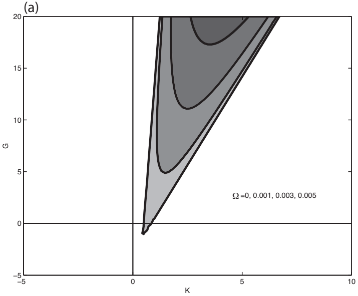
- (a) Analytic model with $\beta_0 = 0.025,\ 0.075,\ 0.15$, and 0.225. The left boundary is vertical at $\beta_0 = 0.101 = \beta_{rp,iw}$.
- (b) Full MHD with $S = 10^6$ with $\beta_0 = 0.07, 0.09,\ 0.11 = \beta_{rp,iw}$, and $\beta_0 = 0.13$, and the left boundary is indeed vertical at $\beta_{rp,iw}$.
- (c) Full MHD with $S = 10^5$ with the same β_0 values as (b) and the left boundary is also vertical as $\beta_{rp,iw} = 0.12$ is crossed.
- Qualtivative structure of the maps is captured by reduced MHD (a). Both results have vertical line at $\beta_{rp,iw}$. Effects of Ω, K_i and G_i change above $\beta_{rp,iw}$,where Ω becomes destabilizing.

Results qualitatively similar between Analytic and Full MHD models for $\beta < \beta_{rp,iw}$: Increasing Ω Stabilizing



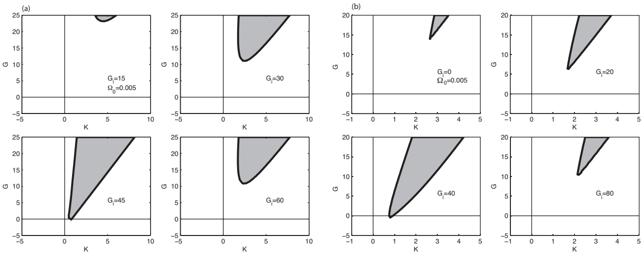
- (a) Analytic with $\beta_0 = 0.068 < \beta_{rp,iw} = 0.101$.
- (b) Full MHD with $\beta_0 = 0.09 < \beta_{rp,iw} = 0.12$.
- The results show that increasing Ω increases the stable area for $\beta_0 < \beta_{rp,iw}$ except for small Ω.

Results: $\beta > \beta_{rp,iw}$ where Ω is Destabilizing; analytic and numerical



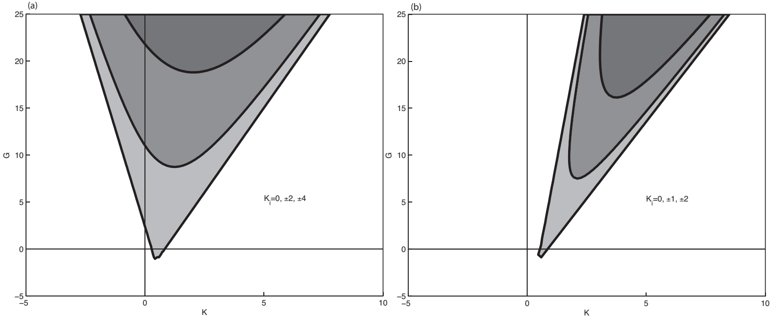
- $S = 10^5$, $G_i = K_i = 0$ and varying rotation Ω.
- (a) simplified model with $\beta_0 = 0.12$ (b) full MHD model with $\beta_0 = 0.13$.
- The plasma Doppler shift frequencies in (a) and (b) are $\Omega = 0, 0.001, 0.003, 0.005$.
- These results show that for $\beta_0 > \beta_{rp,iw}$ the stable region shrinks as $|\Omega|$ increases.

Given a particular Ω, Gi can optimally counter and have the largest stable window equivalent to Ω=0; analytic and numerical



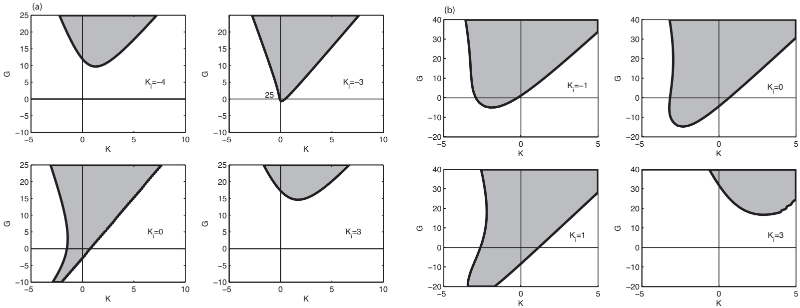
- In (a) we have reduced MHD same as above except for the wall time, which is made equal to the numerical case, $\tau_w = 2 \times 10^4$, and $G_i = 15, 30, 45, 60$.
- In (b) we have full MHD with parameters same as above, with $G_i = 0,\ 20,\ 40,\ 80$.
- There is an optimal value of G_i; for this value the effective wall rotation rate Ω_w is equal to Ω and the stable region is maximized. G_i equivalent to Ω_w.

Analytic Results: With $\Omega = 0$, finite K_i is destabilizing for $\beta < \beta_{rp,iw}$ and $\beta > \beta_{rp,iw}$



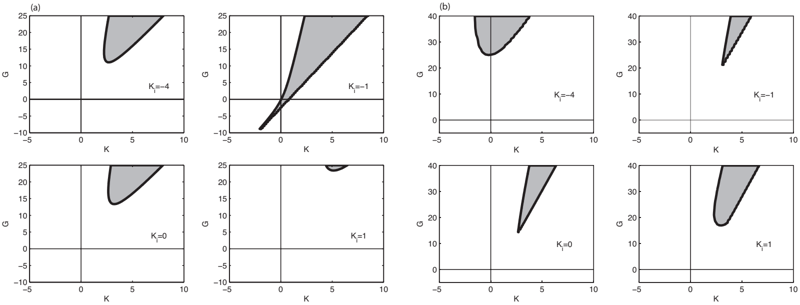
- Stability diagrams for the reduced MHD model only
- (a) $\beta_0 = 0.068 < \beta_{rp,iw}$ with $K_i = 0, \pm 2, \pm 4$
- (b) $\beta_0 = 0.15 > \beta_{rp,iw}$ with $K_i = 0, \pm 1, \pm 2$.
- In both regimes of β_0, K_i decreases the size of the stable region.

With $G_i = 0, \Omega = 0.005$, K_i is destabilizing for $\beta < \beta_{rp,iw}$; analytic and numerical



- (a) Reduced MHD model with $\beta_0 = 0.068$
- (b) the full MHD model with $\beta_0 = 0.09$.
- Optimal value of $|K_i|$ is small and larger values destabilize in the $\beta_0 < \beta_{rp,iw}$ regime.

With $G_i = 0$, $\Omega = 0.005$, K_i has an optimal value for $\beta > \beta_{rp,iw}$; analytic and numerical

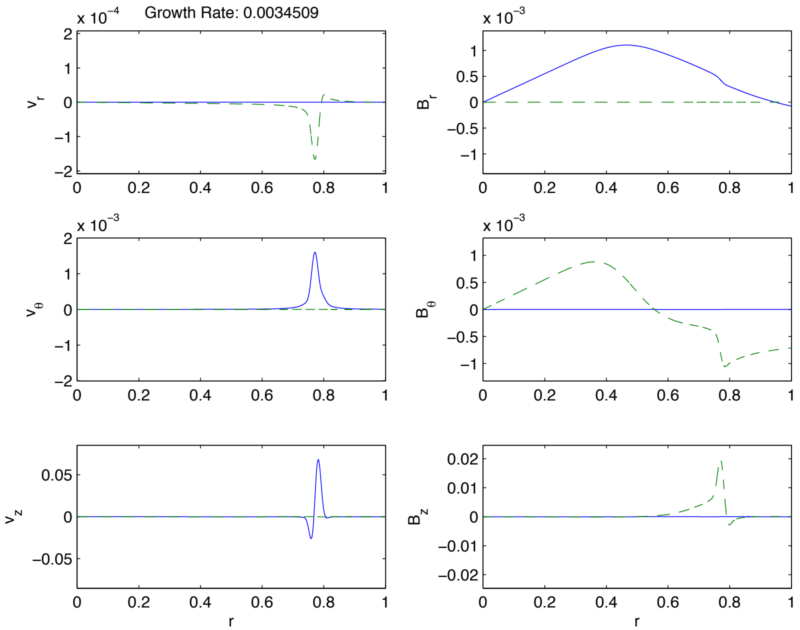


- (a) Reduced MHD model with $\beta_0 = 0.15$
- (b) Full MHD model with $\beta_0 = 0.13$
- In (a) the optimal value of K_i is -1. In (b) the stability regions are more complex, but optimal for $|K_i|$ small.

Summary

- Feedback with complex gain G multiplying normal component of $\tilde{\mathsf{B}}$ and complex gain K multiplying tangential component. G_i and K_i represent simple phase shift of coils.
- Full resistive MHD model agrees with reduced resistive MHD model using stepfunction profiles
- For $\beta < \beta_{rp,iw}$ rotation Ω and $G_i \sim \Omega$ stabilize, as expected. K_i stabilizes in different way
- For $\beta > \beta_{rp,iw}$ rotation Ω and G_i ***destabilize***. K_i destabilizes too
- In $\beta > \beta_{rp,iw}$ regime with Ω: can optimize the feedback stable region by applying G_i such that $\Omega_w = \Omega$. There is an optimal K_i too, but no obvious equivalence

Example Eigenfunction shows layer response



Example Eigenfunction shows layer response